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# On a remarkable similarity between the propagation of whistlers and the bow wave of a ship

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**Abstract.** It is well known that lightning-generated whistlers propagate along the Earth's magnetic field lines within a cone that at low frequencies makes an angle of 19°28′ with respect to the local magnetic field. This angle turns out to be exactly the same as the half-angle of the bow wave of a ship in deep water. Both problems are complicated by the fact that the wave propagation is dispersive. In this paper we show that these two seemingly unrelated problems can be understood using the same basic approach, which is to analyze the direction of the group velocity as a function of the wave normal angle. This approach may have applications to other problems of geophysical interest, such as the bow wave generated by the interaction of a large object with a moving plasma.

## Introduction

Whistlers are audio frequency electromagnetic waves produced by lightning that have a characteristic whistling sound, usually consisting of a tone that decreases rapidly in frequency over a few seconds. Whistlers were discovered during World War I by Barkhausen [1919] using a simple audio frequency radio receiver. The dispersive properties of these signals were later studied by Eckersley [1935], who showed that the frequency varied according to the relation $t = D/\sqrt{f}$, where D is a constant known as the dispersion. Although it was suspected as early as 1930 that the whistling tones originated from lightning [Barkhausen, 1930], it was not until the early 1950s that Storey [1953] was able to develop a theory that explained the long duration of the signals. Storey showed that the durations could be accounted for if the signals travelled along the Earth's magnetic field from one hemisphere to the other in a mode of propagation now known as the whistler mode. This mode of propagation required the existence of an ionized gas in the upper levels of the Earth's atmosphere, a region now known as the magnetosphere. Because of the anisotropy introduced by the magnetic field, Storey was able to show that the wave energy is confined within a cone that makes an angle of $19°28' = \tan^{-1}(1/\sqrt{8})$ with respect to the local magnetic field (see Figure 1).

When a ship moves through the water, a complicated wave pattern is produced downstream of the ship (see Figure 2). The forward envelope of this wave pattern is called the bow wave. In shallow water, where all wavelengths propagate at the same speed, the analysis of the bow wave is relatively simple. However, as the depth of the water increases the problem becomes much more difficult. The propagation becomes dispersive: long wavelengths propagate faster than short wavelengths. The wave pattern produced by a ship in deep water is a classical problem in hydrodynamics that was first solved by Sir William Thomson (Lord Kelvin) in 1887 (see W. Thomson [1891, 1910] and Havelock [1908]). One of the curious features is that in deep water the angle of the bow wave becomes completely independent of the speed of the ship. Kelvin was able to show that the half-angle of the bow wave is 19°28′, exactly the same as the half-angle of the whistler propagation cone. The fact that the whistler mode propagation cone and the bow wave of a ship have exactly the same angle suggests that these two seemingly unrelated problems have a close similarity. The purpose of this paper is to examine the underlying causes of this similarity.



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## The Whistler Mode

To explore the similarities between the propagation of whistlers and the bow wave of a ship, it is useful to give a brief review of the whistler mode. The whistler mode is a right-hand polarized electromagnetic mode that propagates at frequencies below the electron cyclotron frequency, $\omega_c$, and the electron plasma frequency, $\omega_p$. In the region of the Earth's magnetosphere where whistlers propagate, Storey [1953] showed that the index of refraction, n, is given to a good approximation by the equation

$$n^2 = \frac{\omega_p^2}{\omega\ \omega_c \cos\theta} \quad , \tag{1}$$

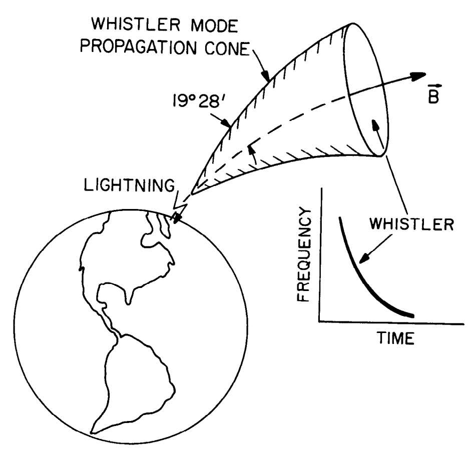


**Figure 1.** Whistlers are electromagnetic waves produced by lightning that propagate through the Earth's magnetosphere at frequencies below the electron cyclotron frequency and plasma frequency. At low frequencies the energy of these waves propagate within a cone that makes an angle of 19°28′ with respect to the local magnetic field.

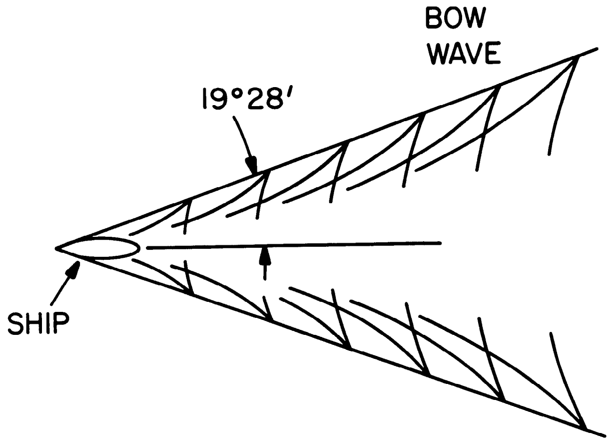


**Figure 2.** A ship moving through the water produces a V-shaped pattern called the bow wave. In deep water, the half-angle of the bow wave is independent of the speed of the ship and has a value of 19°28′, exactly the same as the half-angle of the whistler propagation cone (see Figure 1).

where $\theta$, the wave normal angle, is the angle between the propagation vector **k** and the magnetic field **B**. This equation is sometimes called the longitudinal approximation and is valid in the limit of low frequencies ($\omega \ll \omega_c \cos\theta$), and high densities ($\omega_p^2 \gg \omega\omega_c$). For a discussion of the region of validity of Equation 1, see Helliwell [1965].

As is well known, in an anisotropic medium the wave energy associated with a narrowband wave packet of wave vector $\vec{k}$ propagates at the group velocity, which is given by $\mathbf{V}_g = \nabla_k \omega(\mathbf{k})$, where $\nabla_k$ is a gradient in **k** space and $\omega(\mathbf{k})$ is the dispersion relation [Stix, 1962]. For the whistler mode, the dispersion relation can be obtained from Equation 1 using the relation $n = ck/\omega$, which gives

$$\omega = \frac{c^2\omega_c}{\omega_p^2} k^2 \cos\theta \quad , \tag{2}$$

where c is the speed of light. To compute the group velocity, it is convenient to resolve the **k** vector into components parallel, $k_\parallel$, and perpendicular, $k_\perp$, to the magnetic field. The group velocity can then be computed from Equation (2) using $\mathbf{V}_g = \hat{a}_\parallel(\partial\omega/\partial k_\parallel) + \hat{a}_\perp(\partial\omega/\partial k_\perp)$, where $\hat{a}_\parallel$ and $\hat{a}_\perp$ are unit vectors parallel and perpendicular to the magnetic field. Using this equation it is easy to show that the group velocity is given by

$$\mathbf{V}_g = \frac{c^2\omega_c k}{\omega_p^2}\left[\hat{a}_\parallel(1+\cos^2\theta) + \hat{a}_\perp \sin\theta\cos\theta\right] \quad . \tag{3}$$

To analyze the guiding of whistler mode energy along the magnetic field, it is convenient to define an angle $\Psi$, which is the angle between the group velocity and the magnetic field (see the sketch at the top of Figure 3). From Equation 3, it is evident that the angle $\Psi$ is given by

$$\tan\Psi = \frac{\sin\theta\cos\theta}{1+\cos^2\theta} \quad . \tag{4}$$

A plot of this function is shown at the bottom of Figure 3. Note that as $\theta$ increases, the angle $\Psi$ initially increases, reaches a maximum $\Psi_{max}$ at a critical angle $\theta_c$, and then decreases back to zero. From Equation 4 it is easy to show that $\Psi_{max} = \tan^{-1}(1\sqrt{8}) = 19°28'$.

An impulsive source such as lightning emits a broad spectrum of frequencies and wave vectors. Although the spectrum is broad, the received signal can be filtered into narrowband components, so that the group velocity applies to the packet responsible for each component. The above analysis shows that the wave energy associated with each component is confined within a cone that makes an angle of 19°28′ with respect to the magnetic field. The exact distribution of wave energy within this cone depends on the wave normal distribution at the source. If the wave normal distribution includes waves with wave normal angles around $\theta_c$, where $\partial\Psi/\partial\theta$ is small, then the wave energy is strongly enhanced near and slightly below $\Psi_{max}$. In geometric optics this enhancement is called a caustic focus.

## The Bow Wave of a Ship

The analysis of the bow wave of a ship in deep water is complicated by the fact that long wavelengths propagate faster than short wavelengths. As shown by Lamb [1930] and others, the dispersion relation for waves in deep water is given by $\omega = \sqrt{gk}$, where g is the acceleration of gravity. In Kelvin's 1887 analysis he used a Green's function approach. In this approach the first step is to compute the waveform produced by an impulsive point disturbance. The wave pattern of a ship is then obtained by superposing an infinite series of such impulsive disturbances, the centers of which move along with the ship. This analysis effectively assumes that the pressure disturbance produced by the ship is a delta function, $\delta(\mathbf{r} - \mathbf{V}t)$, where **V** is the velocity of the ship. The integrals involved in the Green's function approach are formidable and must be evaluated using the stationary phase approximation. Although this approach has the advantage of giving the entire wave pattern downstream of the ship, it also has the disadvantage of being quite complicated, which tends to obscure the physical reasons for the processes involved.

Kelvin and others understood that the group velocity is intimately involved in the bow wave problem. The key to

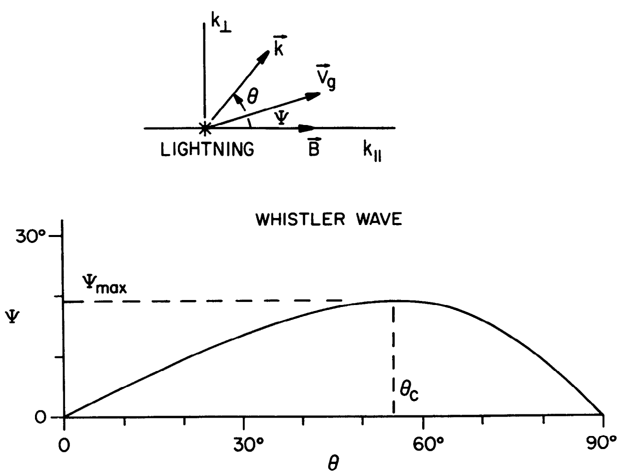


**Figure 3.** The angle of the whistler propagation cone can be obtained by plotting the group velocity direction, $\Psi$, as a function of the wave normal angle, $\theta$. The maximum group velocity angle, $\Psi_{max} = 19°28'$, occurs at $\theta_c = 54°44'$.

a simple understanding of the bow wave is to regard the ship as an isotropic source of waves and to compute the direction of energy flow (i.e., the group velocity) as a function of the wave normal direction. The problem is made particularly simple, and can be seen to be closely analogous to the whistler problem, if the calculations are carried out in the frame of reference of the ship. In the ship frame of reference, the motion of the water introduces an anisotropy, very similar to the anisotropy introduced by the magnetic field in the whistler problem. To compute the dispersion relation $\omega'(\mathbf{k})$ in the ship frame of reference it is necessary to take into account the Doppler shift $\mathbf{k}\cdot\mathbf{V}$, so that $\omega' = \omega - \mathbf{k}\cdot\mathbf{V}$, where $\omega$ is the frequency in the rest frame of the water. Using $\omega = \sqrt{gk}$, the dispersion relation becomes $\omega' = \sqrt{gk} - \mathbf{k}\cdot\mathbf{V}$. It is now easy to compute the group velocity. Using the same $\parallel$ and $\perp$ notation as in the previous section (but now relative to the velocity of the ship) the group velocity becomes

$$\mathbf{V_g} = \hat{a}_\parallel\left(-V + \frac{1}{2}\sqrt{\frac{g}{k}}\cos\theta\right) + \hat{a}_\perp\frac{1}{2}\sqrt{\frac{g}{k}}\sin\theta) \quad . \quad (5)$$

By itself Equation 5 does not completely determine the group velocity. To complete the analysis one must consider an additional factor that has no counterpart in the whistler problem. Since the analysis is being carried out in $\mathbf{k}$ space the source term $\delta(\mathbf{r}-\mathbf{V}t)$ must be Fourier transformed, which becomes $\delta(\omega-\mathbf{k}\cdot\mathbf{V})$. The delta function implies that the emitted wave distribution must satisfy $\omega = kV\cos\theta$. This constraint is analogous to the Cerenkov condition often encountered in charged particle radiation problems [Jackson, 1962]. Combining the Cerenkov condition with $\omega = \sqrt{gk}$ then gives $\sqrt{g/k} = V\cos\theta$. Using this relation one can eliminate $\sqrt{g/k}$ from Equation 5 so that the group velocity is given by

$$\mathbf{V_g} = \frac{1}{2}V[-\hat{a}_\parallel(1 + \sin^2\theta) + \hat{a}_\perp\sin\theta\cos\theta]. \quad (6)$$

Note that the parallel component of the group velocity is always negative. This means that wave energy is always carried downstream, irrespective of the wave normal direction. This result is consistent with the geometric construction given by Lighthill [1978]. Since the parallel component of the group velocity is always negative, it is convenient to define the group velocity direction by an angle $\Psi$ that is measured from the downstream (-V) direction, as shown by the sketch at the top of Figure 4. From Equation 6 it is easy to see that $\Psi$ is given by

$$\tan\Psi = \frac{\sin\theta\cos\theta}{1 + \sin^2\theta} \quad . \quad (7)$$

A plot of $\Psi$ as a function of $\theta$ is shown at the bottom of Figure 4. As can be seen, this plot has exactly the same shape as for the whistler mode, except that $\theta$ is replaced by the co-angle, $\pi/2-\theta$. This is due to the fact that $\cos^2\theta$ in Equation 4 becomes $\sin^2\theta$ in Equation 7. Since the functional forms are the same, it is obvious that the maximum group velocity angle, $\Psi_{max} = 19°28'$, is exactly the same as for the whistler mode.

As with whistler propagation, the wave intensity is strongly enhanced near and slightly below $\Psi_{max}$ due to the caustic focus effect described earlier. The strong enhancement of the wave amplitude near $\Psi_{max}$ is easily observed for a ship moving through a quiet lake or river. The exact distribution of energy within the envelope of allowed group velocity directions depends on the shape and the size of the hull, which determines the wave normal distribution and the wavelengths that are most strongly excited.

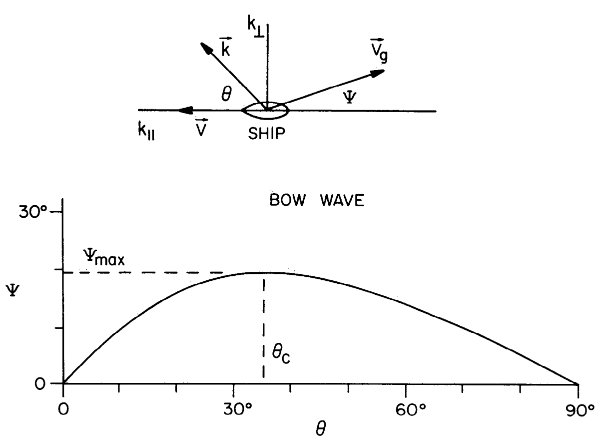


Figure 4. The angle of the bow wave of a ship can be obtained by plotting the group velocity direction, $\Psi$, in the ship frame of reference as a function of the wave normal angle, $\theta$. The $\Psi(\theta)$ plot has exactly the same shape as for the whistler mode, except that $\theta$ is replaced by $\pi/2$ - $\theta$. The maximum group velocity angle, $\Psi_{max} = 19°28'$, occurs at $\theta_c = 35°16'$.

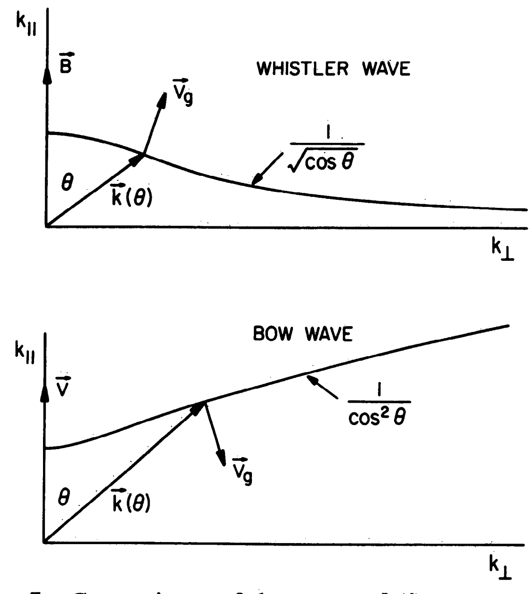


Figure 5. Comparisons of the curves, $\mathbf{k}(\theta)$, generated by plotting the propagation vector as a function of the wave normal direction. The group velocity, $V_g$, is always perpendicular to these curves. Note that in both cases the group velocity is always aligned near the $k_\parallel$ axis.

## Discussion

It is apparent from the preceding analyses that the underlying similarity between the propagation of whistlers and the bow wave of a ship has to do with the group velocity. From very general principles, it can be shown that the group velocity is perpendicular to the curve, $\mathbf{k}(\theta)$, generated by plotting the propagation vector as a function of the wave normal angle [Stix, 1962]. It follows then that the shape of the $\mathbf{k}(\theta)$ curve has a fundamental effect on the anisotropic nature of the propagation. From Equation 1, with $k = \omega n/c$, it is easy to see that $\mathbf{k}(\theta)$ for the whistler mode varies as $1/\sqrt{\cos\theta}$. For the bow wave of a ship, the corresponding result can be obtained by noting that the Cerenkov condition requires that $\omega = kV \cos\theta$. Using $\omega = \sqrt{kg}$ it is then easy to show that $\mathbf{k}(\theta)$ varies as $1/\cos^2\theta$. Corresponding plots of $\mathbf{k}(\theta)$ for both types of waves are shown in Figure 5. The group velocity directions, which are perpendicular to the $\mathbf{k}(\theta)$ curves, are also shown.

As can be seen from Figure 5, the $\mathbf{k}(\theta)$ curves for the two modes of propagation have different shapes. This is not surprising since the two problems do not have the same dispersion relation. However, the two $\mathbf{k}(\theta)$ curves have some very similar features. Both curves increase monotonically from a finite value at $\theta = 0$ to infinity at $\theta = \pi/2$, with a single inflection point, where the sign of the curvature reverses. Since the group velocity is always perpendicular to $\mathbf{k}(\theta)$, it is this inflection point that constrains the group velocity to remain within a narrow range of directions around the $k_\parallel$ axis. It is also evident that the inflection point is a direct consequence of the infinity at $\theta = \pi/2$, a condition often called a resonance cone in the plasma literature [see Stix, 1962]. If a resonance cone did not exist, then all group velocity directions would be possible and no limiting angle would occur.

Although the existence of a resonance cone at $\theta = \pi/2$ assures that the wave energy is constrained within a narrow range of directions around the $k_\parallel$ axis, the exact limits depend on the detailed shape of $\mathbf{k}(\theta)$. That the limiting angles are exactly the same for both the whistler mode and the bow wave is a mathematical peculiarity of $\mathbf{k}(\theta)$ functions of the form $1/\cos^m\theta$. It is easy to show that the limiting angle of the group velocity for functions of this form is given by

$$\tan\Psi_{max} = \frac{1}{2}\left|\frac{1}{\sqrt{m}} - \sqrt{m}\right| \quad . \tag{8}$$

For the special case $m = 1$, the angle $\Psi_{max}$ is exactly zero. It is easily verified that this case corresponds to the shear Alfven wave [Stix, 1962]. A well-known property of the shear Alfven wave is that the wave energy is guided exactly along the magnetic field. As m increases either above or below one, the angle $\Psi_{max}$ increases. Note that m values with an inverse relationship (i.e., 1/2 and 2, 1/3 and 3, etc.) always have the same limiting group velocity angle. Thus, $m = 1/2$, which corresponds to the whistler mode, and $m = 2$, which corresponds to the bow wave, have the same limiting angle.

Although the above results are largely of an instructional nature, the basic approach may have useful applications to other problems of geophysical interest. For example, during the recent Galileo flyby of the asteroid Gaspra, Kivelson et al. [1993] observed a series of magnetic field perturbations that are believed to be due to a whistler mode bow wave produced by the interaction of the asteroid with the solar wind. The approach used here of first computing the dispersion relation in the rest frame of the object, including the Cerenkov condition, and then computing the limiting angle of the group velocity may be useful for analyzing interactions of this type.

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