Determination of Hydrogen Ion Concentration, Electron Density, and Proton Gyrofrequency from the Dispersion of Proton Whistlers

DONALD A. GURNETT AND STANLEY D. SHAWHAN[1]

Department of Physics and Astronomy, University of Iowa, Iowa City

Abstract. In this paper we discuss a method for determining H^+ concentration, electron number density, and proton gyrofrequency in the vicinity of a satellite by measurements of the asymptotic frequency-time profile of a proton whistler near the proton gyrofrequency. This new technique is applied to proton whistlers received by the Injun 3 VLF receiver. The calculated values of H^+ concentration and electron density are shown to be in good agreement with measurements by other experimenters at similar altitudes, latitudes, and local times. B values calculated from the proton gyrofrequency are compared with values calculated from the Jensen and Cain expansion for the geomagnetic field. It is shown that the wave energy of a proton whistler is guided very nearly along the geomagnetic field and that the parallel component of the group velocity is closely approximated by the group velocity for longitudinal propagation. It is found that for frequencies near the proton gyrofrequency at the satellite the kernel multiplying $n(H^+)^{1/2}$ in the travel-time integral is large only in the region near the satellite. Assuming that the H^+ concentration is uniform within this region, an expression is derived for the travel time of a proton whistler near the proton gyrofrequency. The H^+ concentration and proton gyrofrequency are obtained by fitting this theoretical frequency-time expression to observed proton whistler signals. By combining this method for determining $n(H^+)$ with the crossover frequency method for determining $\alpha_1 = n(H^+)/n_e$ the electron density can also be determined.

1. INTRODUCTION

In the VLF recordings from the Injun 3 and Alouette 1 satellites, an unusual ion gyrofrequency phenomenon was observed following the reception of short fractional-hop whistlers [*Smith et al.,* 1964]. The new effect appears as a tone which starts immediately after the reception of a short fractional-hop whistler and initially shows a rapid rise in frequency, asymptotically approaching the gyrofrequency for protons in the plasma surrounding the satellite (see Figure 1). The phenomenon was explained [*Gurnett et al.,* 1965; and *Gurnett,* 1965] as simply a dispersed form of the original lightning impulse, the dispersion arising from the effects of ions on the propagation. The characteristic tone occurring immediately after the reception of a whistler was identified as a left-hand-polarized ion cyclotron wave and is called a proton whistler. The occurrence of proton whistlers in the Injun 3 data is discussed in detail by *Shawhan* [1966].

At a frequency denoted by ω_{12}, the proton whistler and electron whistler frequency-time traces are coincident in time (see Figure 1). This frequency was identified as the crossover frequency, a characteristic frequency of the plasma surrounding the satellite. The abrupt change in the group velocity near ω_{12} occurs as the wave polarization changes sign at the crossover frequency. Measurements of the crossover frequency using proton whistlers received by Injun 3 have been used to provide accurate measurements of the fractional concentration $\alpha_1 = n(H^+)/n_e$ in the ionosphere [*Shawhan and Gurnett,* 1966].

In this paper we discuss a method of determining the H^+ concentration, electron number density, and proton gyrofrequency in the vicinity of the satellite by measurements of the travel time of a proton whistler for frequencies near the proton gyrofrequency. This technique is applied to proton whistlers received by Injun 3. The calculated values of H^+ concentration and electron density are shown to be in good agreement with measurements by other experimenters at similar altitudes and local times.

[1] National Aeronautics and Space Administration graduate trainee.

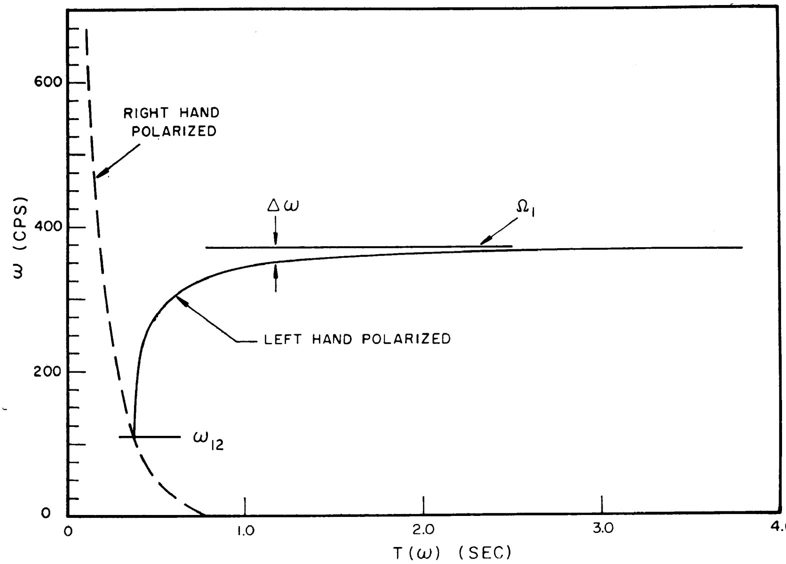


Fig. 1. Calculated proton whistler travel time for the model ionosphere shown in Figure 5.

Values of the geomagnetic field computed by this proton whistler method are compared with values obtained from the Jensen and Cain spherical harmonic expansion of the geomagnetic field for epoch 1960.

This new method of determining the local H^+ concentration, electron density, and proton gyrofrequency is based on the theoretical explanation of the proton whistler. As discussed by *Gurnett et al.* [1965] and *Gurnett* [1965], the proton whistler is a left-hand-polarized ion cyclotron wave propagating in the frequency band available just below the proton gyrofrequency. For frequencies near an ion gyrofrequency the dispersion relation for the ion cyclotron wave becomes particularly simple; it is discussed in section 2 of this paper. It is shown that the wave energy of a proton whistler is guided very nearly along the geomagnetic field and that the group velocity is nearly independent of the wave normal angle and can be closely approximated by the group velocity for longitudinal propagation. The group travel time of proton whistlers observed at the satellite is given by a line integral along the ray path from the lightning source to the satellite. It is found that for frequencies near the proton gyrofrequency (at the satellite) the kernel multiplying $n(H^+)^{1/2}$ in the integrand is large only in the region near the satellite. Assuming that the H^+ concentration is approximately uniform within this region, an expression is derived for the travel time of the proton whistler for frequencies near the proton gyrofrequency. The H^+ concentration and proton gyrofrequency in the vicinity of the satellite are obtained by fitting this theoretical travel time expression to observed frequency-time traces for proton whistlers. By combining this method for determining $n(H^+)$ with the crossover frequency method for determining α_1 the electron density n_e can be calculated using $n_e = n(H^+)/\alpha_1$.

2. Group Travel Time near an Ion Gyrofrequency

The group travel time of a proton whistler from the source of the lightning impulse to the satellite is given by the line integral:

$$t(\omega) = \int_p \frac{ds}{u} \tag{1}$$

$$u = |\partial\omega/\partial\mathbf{K}|$$

Before this integral can be evaluated, we must consider the ray path p and the group velocity u along the ray path. Since the group velocity of the left-hand-polarized ion cyclotron wave is much less than the group velocity of the right-hand-polarized wave, we recognize that the major contribution to the travel time integral occurs along the part of the path where the wave is left-hand-polarized. This is especially true for frequencies near an ion gyrofrequency, since the group velocity of the ion cyclotron wave approaches zero at the ion gyrofrequency. Thus, we limit our attention to an ion cyclotron wave with frequencies near an ion gyrofrequency.

In the notation introduced by *Stix* [1962] the refractive index for ion cyclotron waves can be obtained from

$$n^4 \cos^2 \theta - n^2 S(1 + \cos^2 \theta) + RL = 0 \quad (2)$$

$$R = 1 - \sum_k \frac{\pi_k^2}{\omega(\omega + \Omega_k)} \quad (3)$$

$$L = 1 - \sum_k \frac{\pi_k^2}{\omega(\omega - \Omega_k)} \quad (4)$$

$$S = \tfrac{1}{2}(R + L)$$

where subscripts (k) are e, 1, 2, and 3 for electrons, H^+, He^+, and O^+, respectively.

The plasma frequency and gyrofrequency of the kth constituent are π_k and Ω_k (including sign), respectively, and θ is the angle between the wave normal and the static magnetic field. In the derivation of equation 2, it is assumed that $L > 0$, that $\pi_e^2 \gg \Omega_1\Omega_e \gg \omega^2$, and that $\tan^2 \theta \ll \tan^2 \theta_{Res}$ (θ_{Res} is the vertex angle of the wave velocity surface).

For the electron densities and magnetic field strengths found in the ionosphere the inequalities $L > 0$ and $\pi_e^2 \gg \Omega_1\Omega_e \gg \omega^2$ are valid for proton whistlers. The range of wave normal angles for which the inequality $\tan^2 \theta \ll \tan^2 \theta_{Res}$ is satisfied can be seen as follows from the equation for the vertex angle of the wave velocity surface: For frequencies near the kth ion gyrofrequency θ_{Res} is given by

$$\tan^2 \theta_{Res} = (2/\alpha_k)(M_k/M_e)\Delta\omega/\Omega_k$$

$$\alpha_k = n(k)/n_e \qquad \Delta\omega = \Omega_k - \omega \quad (5)$$

$$M_k/M_e = \text{Ion mass/Electron mass}$$

For proton whistlers $\Delta\omega/\Omega_1$ is never found to be less than about 5×10^{-3} because of cyclotron damping for frequencies near Ω_1 [*Gurnett and Brice*, 1965]). Using this minimum value for $\Delta\omega/\Omega_1$, it is evident that θ_{Res} is always near $\pi/2$. Thus, equation 2 is valid for proton whistler propagation in the ionosphere provided that θ is not near $\pi/2$. Wave normal angles near $\pi/2$ are not likely, because the large refractive index in the ionosphere causes the wave normal to be refracted to near vertical so that $\theta = \pi/2$ could only occur near

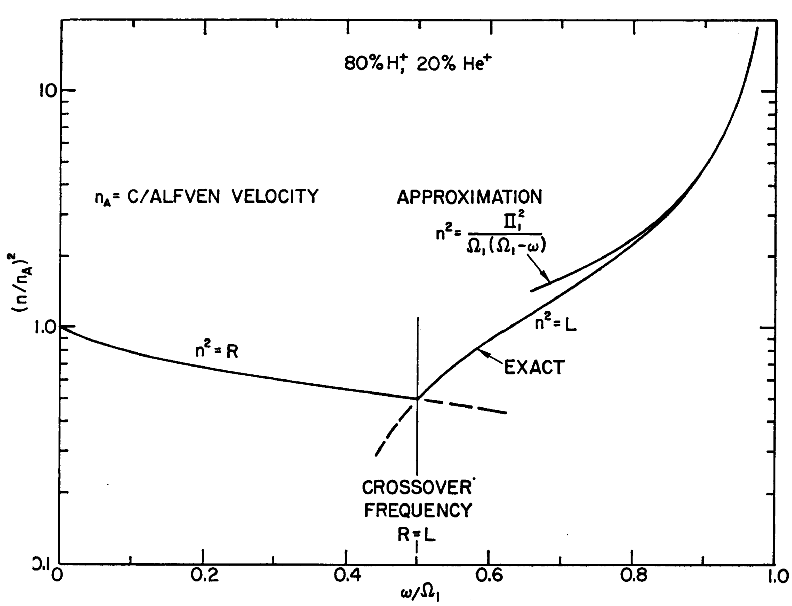


Fig. 2. Refractive index for longitudinal propagation compared with the approximate expression given by equation 7.

the equator. Relatively few proton whistlers are received near the equator [*Shawhan,* 1966] compared with the occurrence rates at mid-latitudes.

To further simplify equation 2, we now restrict the discussion to frequencies near an ion gyrofrequency so that $L \gg R$. With this assumption the solution for the slow wave from equation 2 is

$$n^2 = L(1 + \cos^2\theta)/(2\cos^2\theta) \qquad (6)$$

To obtain an expression for $L(\omega)$ consistent with the assumption that the frequency under discussion is close to an ion gyrofrequency we expand equation 4 about Ω_1 in terms of $\Delta\omega = \Omega_1 - \omega$. Neglecting higher-order terms in the parameter $\Delta\omega/\Omega_1$, we obtain for the refractive index of ion cyclotron waves near Ω_1

$$n^2 = \frac{\pi_1^2}{\Omega_1(\Omega_1 - \omega)} \frac{(1 + \cos^2\theta)}{2\cos^2\theta} \qquad (7)$$

To illustrate the range of frequencies for which this expression is valid the refractive indices for longitudinal propagation, $n^2 = R$ and $n^2 = L$, are plotted in Figure 2. The plasma in this example consists of 80% H^+, 20% He^+, and electrons. The approximate expression given by equation 7 is also compared with the exact equation $n^2 = L$ for $\theta = 0$. Equation 7 is seen to be a good approximation for frequencies well above the crossover frequency.

The phase refractive index surface $n(\theta)$ given by equation 7 is shown in Figure 3. We see that, since the group ray direction is normal to the phase refractive index surface, the ion cyclotron wave energy is constrained to propagate nearly along the static magnetic field. The angle ψ between the group ray direction and $\mathbf{B}_0$ is given by

$$\tan\psi = \frac{\sin\theta\cos^3\theta}{1 + \cos^4\theta} \qquad (8)$$

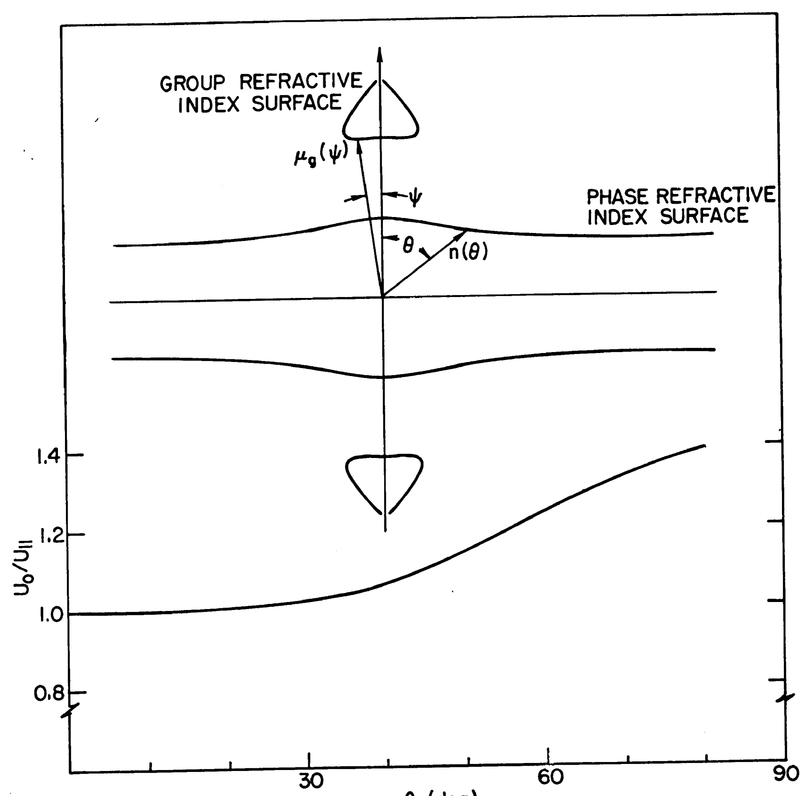


Fig. 3. Phase and group refractive index surfaces for ion cyclotron waves and $u_0/u_{\parallel}$ as a function of wave normal angle.

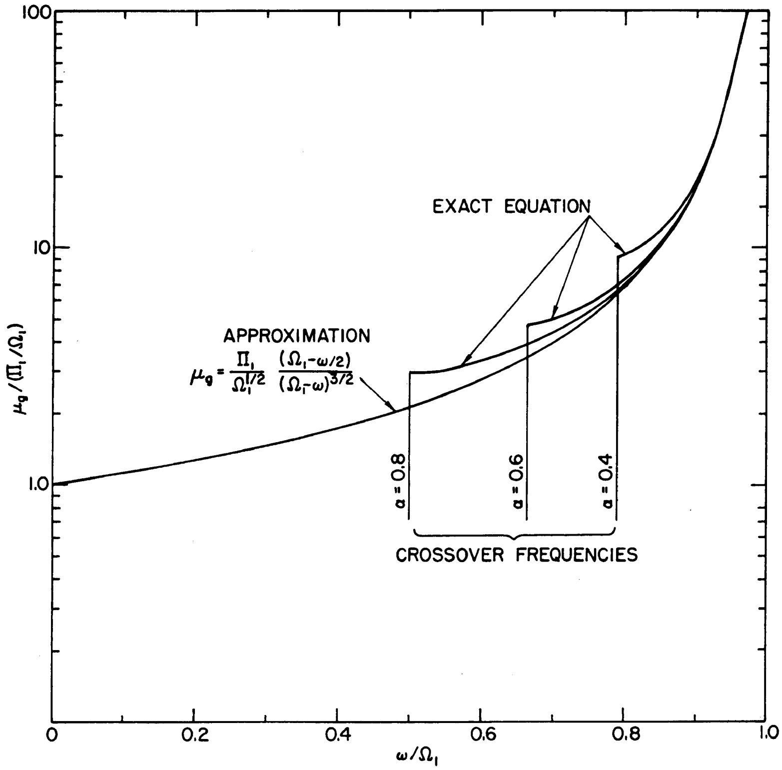


Fig. 4. Group velocity for longitudinal propagation for a H^+, He^+ plasma with $\alpha_1 = 0.8, 0.6$, and 0.4 as compared with the approximate expression given by equation 11.

From this expression it can be shown that $\psi \leq 12.3°$. Thus, the ray path of proton whistlers near the proton gyrofrequency is very nearly along a geomagnetic field line. For comparison, the ray path of an electron whistler (considering only electron motion) was shown by *Storey* [1953] to be confined to angles $\psi \leq 19.3°$.

The group velocity is obtained from equation 7, using

$$|\mathbf{u}| = \frac{c}{n + \partial n/\partial\omega}\left[1 + \left(\frac{1}{n}\frac{\partial n}{\partial\theta}\right)^2\right]^{1/2} \tag{9}$$

We obtain for the component of the group velocity along the static magnetic field ($u_{\parallel}$)

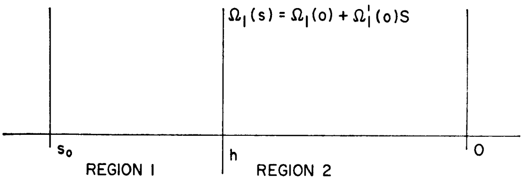
$$u_{\parallel} = u_0 \frac{[2(1 + 2\cos^2\theta + \cos^6\theta)]^{1/2}}{(1 + \cos^2\theta)^{3/2}} \cos\psi \tag{10}$$

$$u_0 = \frac{c\Omega_1^{1/2}(\Omega_1 - \omega)^{3/2}}{\pi_1(\Omega_1 - \omega/2)} \tag{11}$$

These group velocity expressions are plotted in Figure 3 as a group ray refractive index surface $\mu(\psi) = c/|\mathbf{u}|$ and as a plot of $u_0/u_{\parallel}$ versus θ. It is evident from these plots that $u_{\parallel}$ is nearly independent of θ for wave normal angles up to about 50° and is approximated very closely by the group velocity for longitudinal propagation (equation 11).

The group velocity expression for longitudinal propagation u_0 is correct at all frequencies in the limit of a 100% H^+ plasma. For a H^+, He^+ plasma the error in using the approximate equation 11 for the group velocity is shown in Figure 4 by comparing equation 11 with the more complicated but exact expression (equation 4). Evidently this approximate expression is very good for frequencies well above the crossover frequency and rapidly approaches the exact expression as the wave frequency approaches the proton gyrofrequency.

We now consider the travel time integral discussed earlier (equation 1). The ray path is represented schematically as a horizontal axis s with the wave propagating to the right from an impulse source at s_0. The static magnetic field increases monotonically to the left with the proton gyrofrequency having a value $\Omega_1(0)$ at the origin (the observation point). We are only concerned with wave frequencies below and near $\Omega_1(0)$.



It is convenient to divide the propagation path into two regions as shown in the diagram, the total travel time being written $t(\omega) = t_1(\omega) + t_2(\omega)$. In region 1 ($0 \leq s < h$) we assume that the wave is a left-hand-polarized ion cyclotron wave and that the wave frequency is close to the proton gyrofrequency throughout the region. We also assume that the proton gyrofrequency can be approximated by $\Omega_1(s) = \Omega_1(0) + \Omega_1'(0)s$, where $\Omega_1'(0) = \partial\Omega_1/\partial s$. In region 2 ($h < s \leq s_0$), we assume that for the frequencies of interest there are no poles or zeros in the refractive index. Thus, the group velocity along this part of the ray path is always nonzero. (This is equivalent to saying that h is accessible from s_0.) From this we conclude that $t_2(\omega)$ will be a finite continuous function of ω for the frequencies of interest (since $u \neq 0$ anywhere along the path).

In the application of this propagation model to proton whistler propagation in the ionosphere, s_0 is at the base of the ionosphere ($\simeq$100 km), $s = 0$ is the position of the satellite, and the polarization reversal that gives rise to proton whistlers occurs in region 2.

Since the wave in region 1 is an ion cyclotron wave with ω near Ω_1 we know that the ray direction is nearly along the static magnetic field. This allows us to use equation 11 for u in the travel time integral $t_1(\omega)$:

$$t_1(\omega) = \frac{1}{c}\int_h^0 \frac{\pi_1(s)[\Omega_1(s) - \omega/2]\, ds}{\Omega_1(s)^{1/2}[\Omega_1(s) - \omega]^{3/2}} \tag{12}$$

For small $\Delta\omega = \Omega_1(0) - \omega$ the kernel multiplying $\pi_1(s)$ in the integrand is large only in the neighborhood of $s = 0$. We therefore assume that $\pi_1(s)$ varies slowly in region 1 so that we may write (using $\Omega(s) = \Omega_1(0) + \Omega_1'(0)s$):

$$t_1(\omega) = \frac{\pi_1(0)}{c} \cdot \int_h^0 \frac{[\Omega_1(0)+\Omega_1'(0)s-(\omega/2)]\, ds}{[\Omega_1(0)+\Omega_1'(0)s]^{1/2}[\Omega_1(0)-\omega+\Omega_1'(0)s]^{3/2}} \tag{13}$$

We obtain for the integral

$$t_1(\omega) = \frac{\pi_1(0)}{c\Omega_1'(0)}\left[\frac{1}{[1-\omega/\Omega_1(0)]^{1/2}} - \frac{1}{[1-\omega/\Omega_1(h)]^{1/2}} + 2\sinh^{-1}\left[\frac{\Omega_1(0)}{\omega} - 1\right] - 2\sinh^{-1}\left[\frac{\Omega_1(h)}{\omega} - 1\right]\right] \tag{14}$$

Expanding to the first order in terms of the small quantity $\Delta\omega/\Omega_1$ we obtain

$$t_1(\omega) = \frac{\pi_1(0)\Omega_1(0)^{1/2}}{c\Omega_1'(0)}\left[\frac{1}{\Delta\omega}\right]^{1/2} + [\text{Terms} \propto (\Delta\omega)^n] \tag{15}$$

We note that h occurs only in the higher-order terms in $\Delta\omega$; hence, in the limit $\Delta\omega \to 0$ the dependence of $t_1(\omega)$ on h appears only as a constant.

Since $t_2(\omega)$ is finite and continuous about Ω_1 we have, for small $\Delta\omega$, $t_2(\omega) = t_2(\Omega_1) + t_2'(\Omega_1)\Delta\omega$. The asymptotic expression for $t(\omega)$ in the limit $\Delta\omega \to 0$ is dominated by $t_1(\omega)$ and therefore is

$$t(\omega) = \frac{\pi_1(0)\Omega_1(0)^{1/2}}{c\Omega_1'(0)}\left[\frac{1}{\Delta\omega}\right]^{1/2} + \text{Constant} \tag{16}$$

This equation gives the asymptotic form for the long-drawn-out 'high frequency tail' of the proton whistler for ω near the proton gyrofrequency at the satellite (see Figure 1). This relatively simple relation, independent of the concentrations along all but the latter part of the ray path, occurs because of the singular nature of the travel time integral near the proton gyrofrequency.

Equation 16 is the working equation for the determination of $n(H^+)$ and Ω_1 from proton whistlers. The interpretation to be given to equation 16 is as follows:

Assume that the local gyrofrequency at the satellite is Ω_1; then the travel time $t(\omega)$ for the portion of proton whistler signal near the proton gyrofrequency is proportional to the square root of the period of the difference frequency $\Delta\omega = \Omega_1 - \omega$. If $t(\omega)$ versus $(1/\Delta\omega)^{1/2}$ is not found to be a straight line, the assumed gyrofrequency is incorrect and must be adjusted until the best straight-line fit is obtained. This gyrofrequency is then a determination of the true proton gyrofrequency at the satellite .

Since $\partial\Omega_1/\partial s$ can be accurately calculated at the position of the satellite from the geometry of the geomagnetic field, it is evident from equation 16 that the slope of $t(\omega)$ versus $(1/\Delta\omega)^{1/2}$ can be used to obtain $\pi_1(0)$ and hence $n(\mathrm{H}^+)$ from

$$\pi_1^2(0) = 4\pi e^2 n(\mathrm{H}^+)/m_1 \qquad (17)$$

The experimental method used to calculate $\pi_1(0)$ and $\Omega_1(0)$ from proton whistlers and the accuracy of the method are discussed in the next section.

3. Method for Determining $n(\mathrm{H}^+)$ and $\Omega_1(0)$ from the Dispersion of Proton Whistlers

We have shown that equation 16 gives the travel time of a proton whistler in the limit of small $\Delta\omega$. We have also suggested that equation 16 be used to determine $n(\mathrm{H}^+)$ and $\Omega_1(0)$ from the dispersion of proton whistlers received by a satellite. We now discuss a method of calculating $n(\mathrm{H}^+)$ and $\Omega_1(0)$ from a set of measured proton whistler travel times at different frequencies (t_j, ω_j). Since actual measurements must be made for a finite range of $\Delta\omega$ values, and since equation 16 is strictly valid only in the limit $\Delta\omega \rightarrow 0$, we must estimate the error in calculating $n(\mathrm{H}^+)$ and $\Omega_1(0)$ from equation 16 using a realistic range of $\Delta\omega$ values. This we do by testing the accuracy of equation 16 using as data calculated proton whistler travel times for an assumed model ionosphere. By comparing the values of $n(\mathrm{H}^+)$ and $\Omega_1(0)$ for the assumed ionospheric model with the values obtained from fitting equation 16 to the calculated proton whistler travel times we can estimate the errors that arise from using equation 16 for finite values of $\Delta\omega$ (independent of measurement errors). The following is an illustration of one such test case.

In Figure 5 is shown one of several model ionospheres used to test equation 16. The ion-density profiles for this model are calculated us-

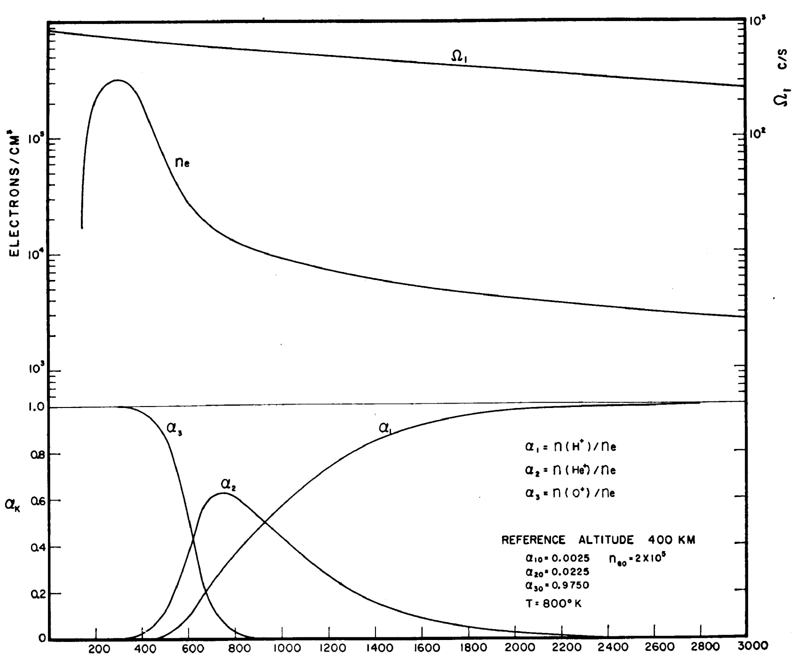


Fig. 5. An ionospheric model used to calculate proton whistler travel times for testing the accuracy of equation 16 when $\Delta\omega$ is finite.

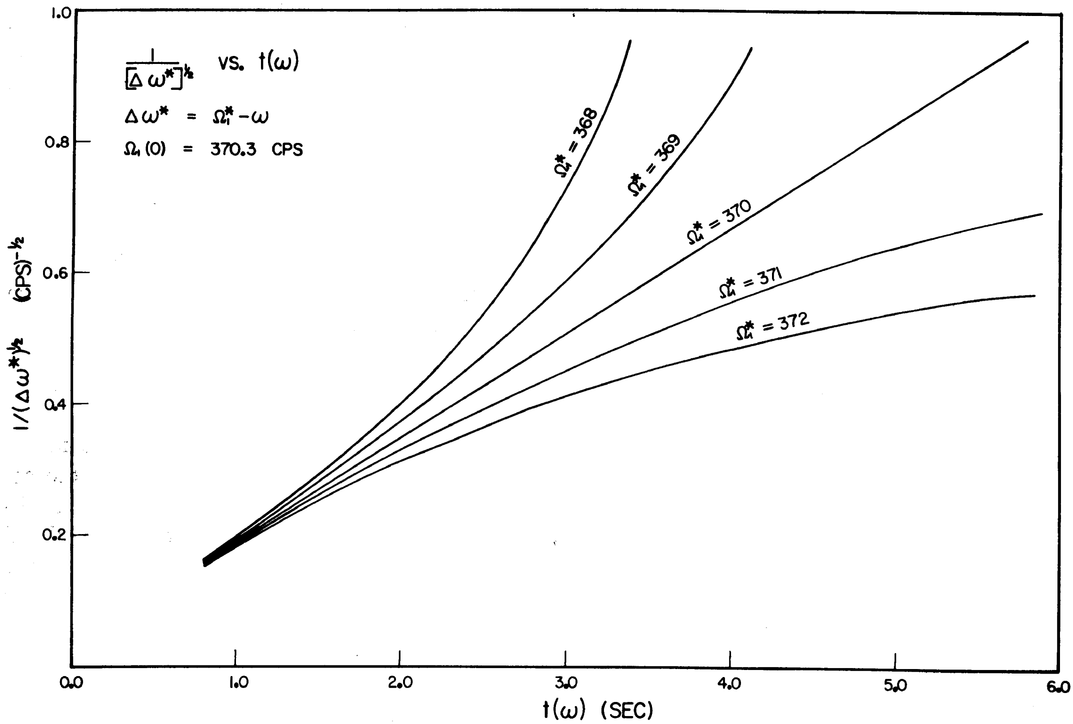


Fig. 6. A plot of $t(\omega)$ versus $[1/\Delta\omega]^{1/2}$ for the calculated proton whistler travel time shown in Figure 1.

ing diffusive equilibrium relations with an ion temperature of 800°K. The detailed choice of the model ionosphere is not crucial, since we only wish to illustrate the method of determining $n(H^+)$ and $\Omega_1(0)$ and to estimate the errors in the method.

Using the ionospheric model of Figure 5, proton whistler travel times $t(\omega)$ to an altitude of 2000 km are calculated by means of the exact equations for longitudinal propagation as discussed by *Gurnett et al.* [1965]; they are shown in Figure 1. In Figure 6 we plot $t(\omega)$ versus $[1/\Delta\omega^*]^{1/2}$, $\Delta\omega^* = \Omega_1^* - \omega$, for this test case, where Ω_1^* is a trial value for the proton gyrofrequency. Five curves are shown for Ω_1^* values differing by 1 cps. The actual proton gyrofrequency at 2000 km for this test case is $\Omega_1(0) = 307.3$ cps. The relation between $t(\omega)$ and $[1/\Delta\omega^*]^{1/2}$ is seen to be very nearly linear when $\Omega_1^* = 307$ cps $\simeq \Omega_1(0)$. However, when Ω_1^* is as little as 1 cps different from $\Omega_1(0)$, the curves deviate markedly from a straight line.

The plots in Figure 6 show that $\Omega_1(0)$ can be determined from proton whistler measurements by finding the value of Ω_1^* for which the measured pairs of points (t_j, $p_j = [1/(\Omega_1^* - \omega_j)]^{1/2}$) lie on a straight line. This value, call it $\hat{\Omega}_1$, is the best estimate of the proton gyrofrequency at the satellite. The H^+ concentration is determined from the slope $S = \Delta t/\Delta p$ (evaluated at $\Omega_1^* = \hat{\Omega}_1$) using the following relation derived from equation 16:

$$n(H^+) = \frac{m_1 C^2}{4\pi e^2} \frac{(\partial\Omega_1/\partial s)^2}{\hat{\Omega}_1(0)} S^2 \tag{18}$$

To implement this scheme for calculating $\Omega_1(0)$ and $n(H^+)$ we use the statistic $T(\Omega_1^*)$ to find the value of Ω_1^* that gives the best-fitting linear relation between the points (t_j, p_j).

$$P(\Omega_1^*) = \frac{\sum_j^n (t_j - \bar{t})(p_j - \bar{p})}{\left[\dfrac{\sum (p_j - \bar{p})^2 \sum (t_j - \bar{t})^2 - [\sum (t_j - \bar{t})(p_j - \bar{p})]^2}{n - 2}\right]^{1/2}} \tag{19}$$

$$\bar{t} = \frac{1}{n} \sum t_i, \quad \bar{p} = \frac{1}{n} \sum p_i$$

The value of T is a maximum when the mean-square error between the experimental points and the best-fit straight line is a minimum [*Lacey*, 1959]. When the experimental points lie exactly in a straight line it is easy to show that the statistical parameter T is infinite.

In Figure 7 is shown a plot of $T(\Omega_1^*)$ versus Ω_1^* for the calculated proton whistler shown in Figure 1. The points (t_j, ω_j) used to calculate $T(\Omega_1^*)$ are 1 cps apart in frequency for $0 < \Delta\omega < 40$ cps. To prevent the pole in p_j at $\Delta\omega^* = 0$ from dominating the sums in equation 19, the points (t_j, p_j) are included in the sums only if $\Delta\omega^* > 1.0$ cps (a justifiable requirement, because cyclotron damping strongly attenuates frequencies for which $\Delta\omega$ is less than about 2 cps). This condition gives rise to small discontinuities in $T(\Omega_1^*)$ as new points are added to the sums in equation 19 (see Figure 7). The sharp peak in $T(\Omega_1^*)$ at $\Omega_1^* = 370.06$ cps indicates that the straight-line fit through the points (t_j, p_j) is the best for this value of Ω_1^*. This conculsion is consistent with the curves shown in Figure 6, which show that the best straight-line fit occurs when Ω_1^* is approximately 370 cps. From the position of the peak in T we estimate the proton gyrofrequency at 2000 km to be 370.06 cps. This gyrofrequency is in error by 0.24 cps (0.05%) from the gyrofrequency used in the model ionosphere. This small error arises because we have used a finite range of $\Delta\omega$ values (40 cps) in equation 16 whereas this equation is strictly valid only in the limit $\Delta\omega \rightarrow 0$. The error is unavoidable inasmuch as actual measurements must be made over a finite range of $\Delta\omega$ values.

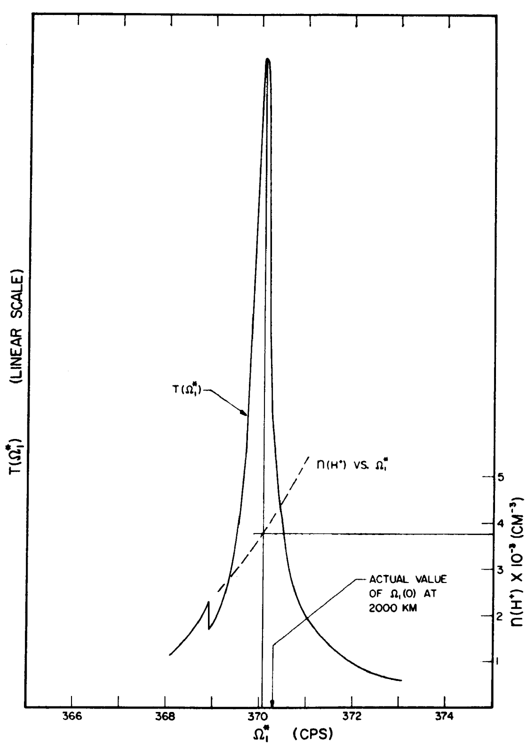


Fig. 7. A plot of $T(\Omega_1^*)$ and $n(H^+)$ as a function of the assumed proton gyrofrequency Ω_1^* for the calculated proton whistler travel time shown in Figure 1.

To calculate $n(H^+)$ from equation 18 the slope of the best-fit straight line through the ponits (t_j, p_j) is calculated by means of

$$S(\Omega_1^*) = \frac{\sum (t_i - \bar{t})(p_i - \bar{p})}{\sum (p_i - \bar{p})^2} \qquad (20)$$

In Figure 7 is shown a plot of $n(H^+)$ versus Ω_1^* for the calculated proton whistler shown in Figure 1. From the peak in $T(\Omega_1^*)$ the H^+ concentration is estimated to be 3.62×10^3 (cm^{-3}). The actual H^+ concentration used in the model ionosphere at 2000-km altitude is 3.76×10^3 (cm^{-3}), an error of 3.8%.

With the ionospheric model shown in Figure 5, test cases similar to the one just described have been analyzed for different altitudes. The error in calculating $n(H^+)$ and $\Omega_1(0)$ for these cases is summarized in Table 1. For all cases the measured frequencies are in the range $0 < \Delta\omega < 40$ cps. Table 1 thus gives an estimate of the errors in determining $\Omega_1(0)$ and $n(H^+)$ from equation 16 when there are no measurement errors. For actual proton whistler data the peak in $T(\Omega_1^*)$ is not nearly as sharp as for these test cases because of experimental errors in measuring the points (t_j, ω_j). To determine error bars we calculate $n(H^+)$ and $\Omega_1(0)$ at the peak of $T(\Omega_1^*)$ for several independently measured data sets (t_j, ω_j) from the same proton whistler. Error bars are given by the standard deviation of these repeated measurements.

Another possible method of fitting equation 16 to observed proton whistler signals is based on maximizing the correlation coefficient be-

TABLE 1

Altitude, km	α_1	Model $\Omega_1(0)$, cps	Calculated $\hat{\Omega}_1(0)$, cps	Error, %	Model $n(H^+)$, cm^{-3}	Calculated $n(H^+)$, cm^{-3}	Error, %
800	0.34	589.5	588.3	0.17	4.25×10^3	3.42×10^3	24
1000	0.54	542.6	541.8	0.15	4.77×10^3	4.24×10^3	11
1200	0.70	500.7	500.2	0.10	4.82×10^3	4.57×10^3	5
1600	0.88	429.0	428.6	0.09	4.33×10^3	4.30×10^3	1
2000	0.94	370.3	370.16	0.05	3.76×10^3	3.62×10^3	4

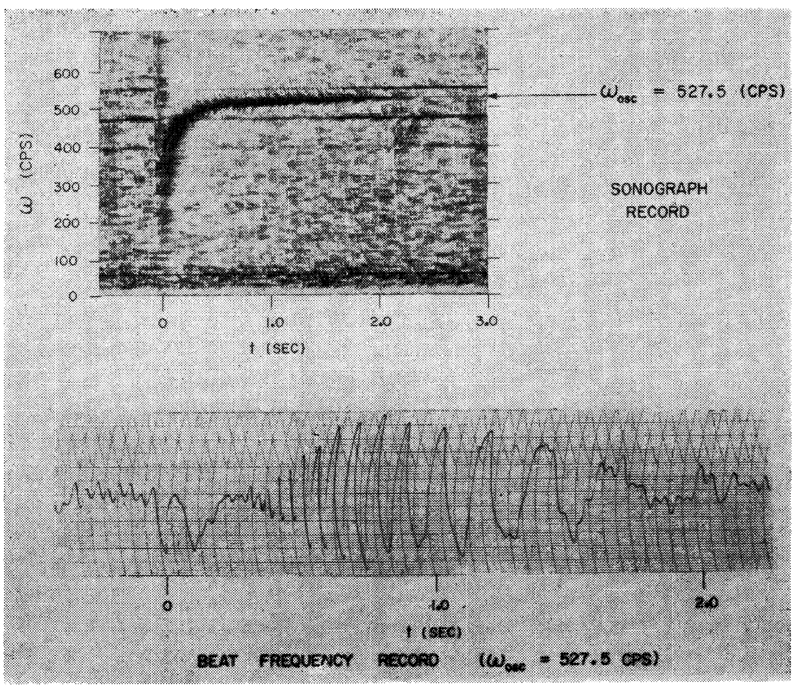


Fig. 8. A sonograph record of a proton whistler signal received by Injun 3, and a beat frequency oscillograph record for the same whistler.

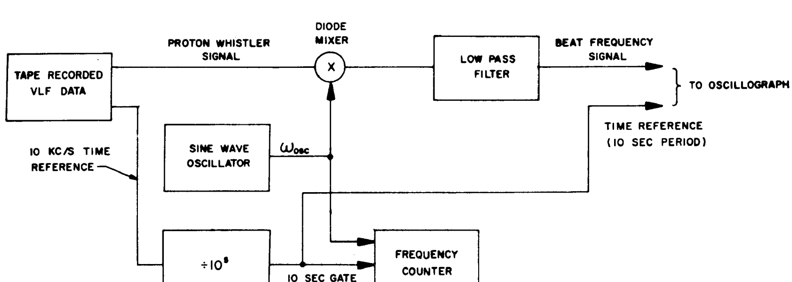


Fig. 9. A block diagram of the equipment for deriving the beat frequency record shown in Figure 8.

tween an ideal signal as predicted by equation 16 and the actual proton whistler signal. It is an extension of a method followed by *Beghin and Siredy* [1964] for analyzing electron whistler travel times. The correlation coefficient is of the following form (suitably normalized):

$$\rho[t_0, n(\mathrm{H}^+), \Omega_1^*] = A \int_{-\infty}^{\infty} S(t) \cos \Phi(t)\, dt$$

$$\Phi(t) = \int \omega(t + t_0)\, dt$$

The function $S(t)$ is the received proton whistler signal, and $\omega(t)$ is obtained by solving equation 16 for ω as a function of t. The correlation coefficient is a function of three unknown quantities: the time origin t_0, the H^+ concentration, and the proton gyrofrequency. Values for these unknown quantities can be determined by finding the value for the three quantities that maximizes the correlation coefficient. As the digitizing and computer programming involved requires a considerable investment in equipment and time, we have not pursued this method.

4. Measurements of $\Omega_1(0)$ and $n(\mathrm{H}^+)$ from Proton Whistlers Received by Injun 3

Up to this time we have assumed that some suitably accurate method exists to measure the travel time of a proton whistler as a function of frequency. Two methods are used: first, points (t_j, ω_j) can be measured directly from spectrograms of proton whistlers; second, the points (t_j, ω_j) can be measured from oscillograph records of the difference frequency signal derived by beating the proton whistler signal against a fixed frequency oscillator. The records obtained with these two methods are contrasted in Figure 8 for a proton whistler received with Injun 3. The spectrogram in this illustration was made on a sonograph frequency-time spectrum analyzer (Kay Electric, Pine Brook, New Jersey). The oscillograph record shown gives the difference frequency between the proton whistler and a fixed frequency oscillator adjusted to a frequency ω_{osc} near the proton gyrofrequency (527.5 cps for the case illustrated). A block diagram of the equipment used to obtain this oscillograph record is shown in Figure 9. A stable 10-kc/s reference tone recorded on the data tape provides the time base for accurately determining the oscillator frequency and putting time markers on the oscillograph record. The bandwidth of the low-pass filter is usually set to about 15 cps. The proton whistler frequency at time t_j is calculated by means of $\omega_j = \omega_{osc} - 1/T_j$ (cps), where T_j is the period of the beat frequency signal at time t_j. Generally measurements from both the sonograph

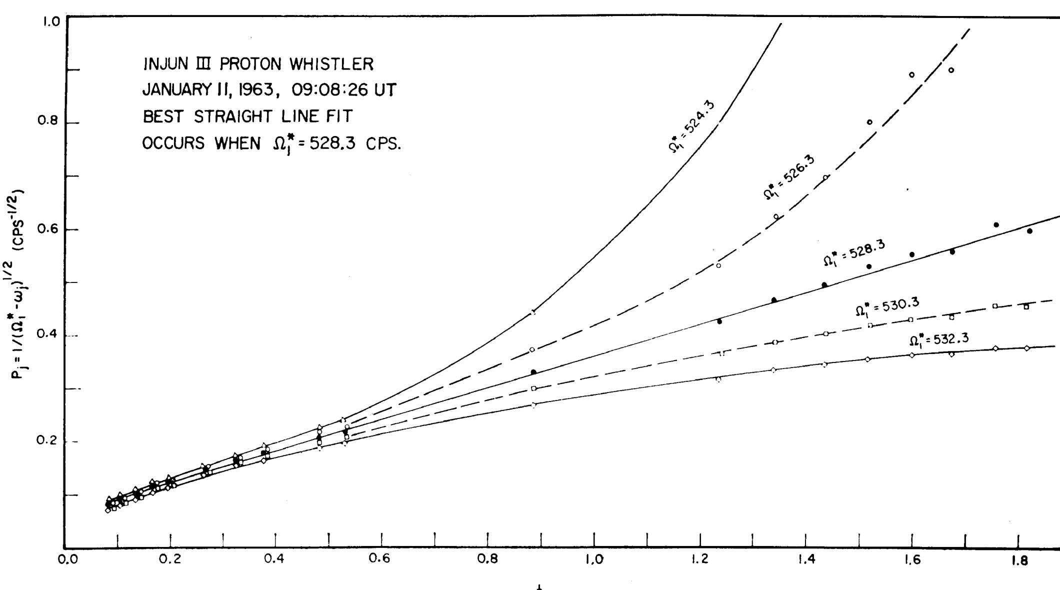


Fig. 10. A plot of $1/(\Omega_1^* - \omega)^{1/2}$ versus t for the proton whistler shown in Figure 8.

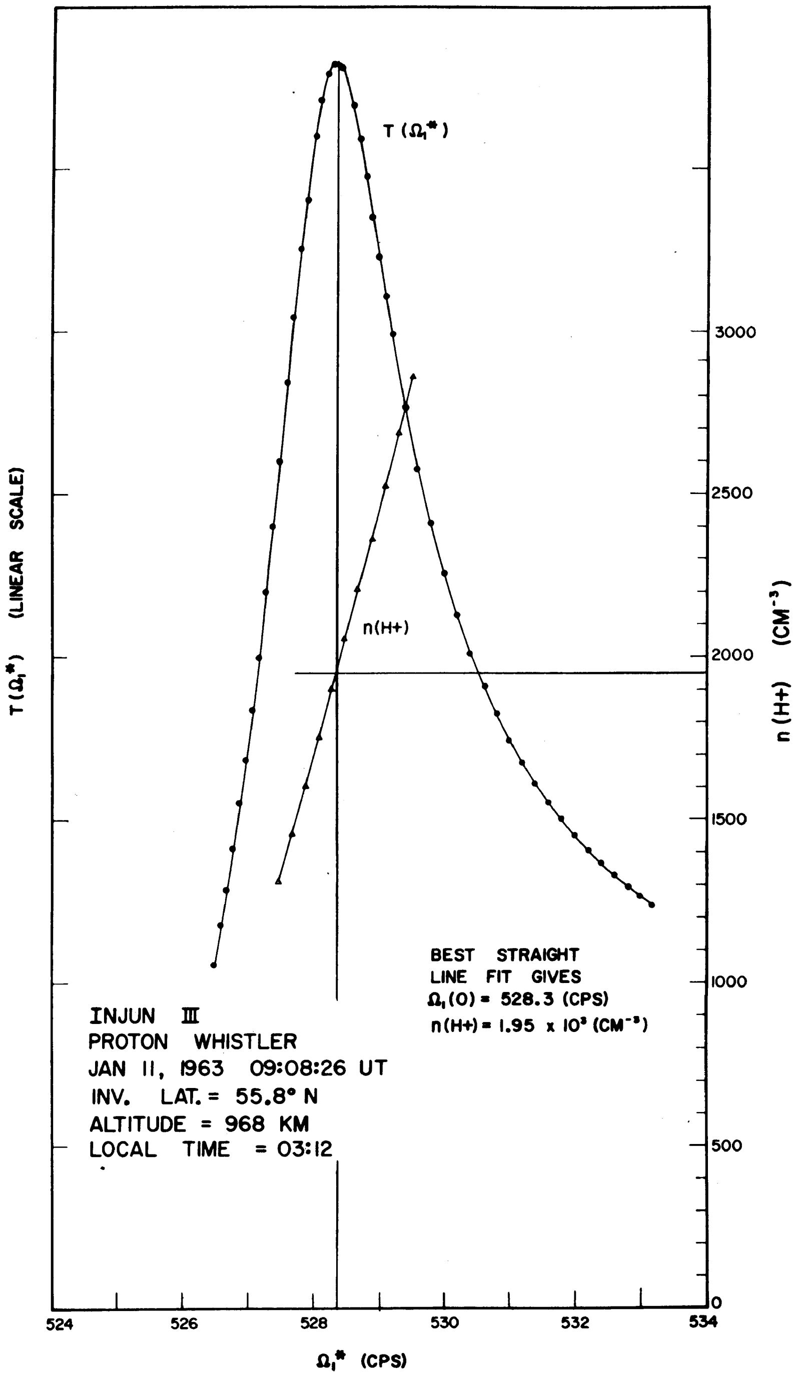


Fig. 11. A plot of $T(\Omega_1{}^*)$ and $n(H^+)$ as a function of the assumed proton gyrofrequency $\Omega_1{}^*$ for the proton whistler shown in Figure 8.

TABLE 2

Satellite Coordinates WN = winter night; SN = summer night					Proton Whistler Determinations			Nearest Alouette Electron Density Data at 1000 Km
UT 1963		Altitude, km	Invariant Latitude, deg	Jensen & Cain Value for $\Omega_1(0)$, cps	$\Omega_1(0)$, cps	$n(H^+) \times 10^{-3}$, cm^{-3}	$n_e \times 10^{-3}$, cm^{-3}	$n_e \times 10^{-3}$, cm^{-3}
Jan. 7 1008:19	WN	1595	40.2	329.2	329.9±0.5	7.93±0.85	9.51±1.00	
Jan. 7 1013:22	WN	1249	50.2	450.0	451.5±0.8	7.72±1.50	13.5±2.5	7.5–12.5
Jan. 11 0908:26	WN	968	55.8	528.2	528.0±0.9	1.90±0.28	2.97±0.50	3.0–7.5
Jan. 11 1059:38	WN	1308	42.4	408.5	412.3±0.7	14.0±2.6	20.4±3.8	10.0–20.0
Jan. 11 1103:12	WN	1058	51.7	516.4	517.9±1.1	5.94±2.58	7.46±3.21	7.5–12.5
Mar. 3 0746:06	WN	1708	48.9	369.0	373.0±0.6	4.80±0.96	5.93±1.05	
June 11 0142:54	SN	1241	41.1	382.0	382.4±0.3	5.46±0.77	9.80±1.50	7.5–12.5
June 11 0147:06	SN	1535	52.1	387.3	387.5±0.6	6.09±1.13	8.01±1.48	

record and the beat frequency record are used. They are complementary in that the beat frequency records give excellent accuracy for wave frequencies near the proton gyrofrequency ($\Delta\omega$ less than about 15 cps), whereas the sonograph records give good accuracy for larger values of $\Delta\omega$ (greater than about 15 cps).

Nineteen points (ω_j, t_j) have been measured for the proton whistler shown in Figure 8. In Figure 10 is shown a plot of $[1/(\Omega_1^* - \omega)]^{1/2}$ versus t for five values of Ω_1^* as calculated from the (ω_j, t_j) measurements. In Figure 11 is shown a plot of the statistical parameter $T(\Omega_1^*)$ for these measurements. The peak in the parameter $T(\Omega_1^*)$ occurs when $\Omega_1^* = 528.3$ cps and $n(H^+) = 1.95 \times 10^3$ (cm^{-3}). These values are the best estimate of $\Omega_1(0)$ and $n(H^+)$ at the satellite from measurements of the oscillograph record in Figure 8. By independently analyzing five independent records for this same proton whistler signal (not all having the same oscillator frequency) the following error bars (1 standard deviation) due to random measurement errors have been determined for it:

$$\Omega_1(0) = 528.0 \pm 0.9(\text{cps})$$

$$n(H^+) = (1.90 \pm 0.28) \times 10^3(\text{cm}^{-3})$$

From the crossover frequency, estimated to be within $278 < \omega_{12} < 330$ (cps), the fractional concentration $\alpha_1 = n(H^+)/n_e$ has been calculated to be (see *Shawhan and Gurnett* [1966]) $0.62 < \alpha_1 < 0.74$ for this proton whistler. From α_1 we obtain the following estimate of the electron number density at the satellite: $n_e = (2.97 \pm 0.50) \times 10^3$ (cm^{-3}).

In Table 2 we present a number of values of $\Omega_1(0)$, $n(H^+)$, and n_e calculated from proton whistlers received by Injun 3. Also shown for comparison are $\Omega_1(0)$ values calculated from the Jensen and Cain expansion for the geomagnetic field at the satellite [*Jensen and Cain*, 1962] and n_e values obtained from the Alouette 1 satellite for the same latitude, season, and local time as the Injun 3 measurements. The Alouette electron-density measurements were kindly provided by M. J. Rycroft of Ames Research Center. The largest disagreement between the proton gyrofrequency determined from proton whistlers and from the Jensen and Cain expansion is 1%. This disagreement we attribute to the accuracy limitations of the Jensen and Cain expansion for the geomagnetic field. The electron densities calculated from proton whistlers are seen to be in reasonably good agreement with electron densities obtained from the Alouette 1 satellite at 1000 km. This general agreement with other data is interpreted as establishing the validity of the proton whistler method as another independent method for the determination of magnetic fields ($\Omega_1(0)$), $n(H^+)$, and n_e in the ionosphere.

5. Conclusion

In this paper the propagation of ion cyclotron waves near an ion gyrofrequency was discussed. It was found that near an ion gyrofrequency the ray path of the left-hand-polarized wave is very nearly along the static magnetic field line and that the parallel component of the group velocity is nearly independent of wave normal angle up to about 50°. A simple equation for the parallel component of the group velocity was given.

These equations were applied to the travel time integral of proton whistlers from the source of the lightning impulse to the satellite. It was found that for frequencies near the proton gyrofrequency at the satellite the kernel multiplying $n(H^+)^{1/2}$ in the integrand is large only in the region near the satellite. Assuming $n(H^+)$ to be approximately uniform within this region, an expression was derived for the frequency-time trace of a proton whistler near the proton gyrofrequency. It was shown that the proton gyrofrequency and the H^+ concentration in the vicinity of the satellite could be obtained by fitting this theoretical frequency-time expression to observed proton whistler signals. By also measuring the crossover frequency of the proton whistler the electron density could be calculated.

This method for determining proton gyrofrequency, H^+ concentration, and electron density was applied to proton whistlers received by the Injun 3 satellite. The calculated values were shown to be in good agreement with measurements by other experimenters at similar altitudes, latitudes, and local times.

This new technique for measuring electron densities is expected to be particularly useful in the VLF studies with the Injun 3 satellite because no other detector was included to determine n_e. This technique illustrates that a considerable amount of information about the composition of the ionosphere can be obtained from a VLF receiver of the type flown on Injun 3. Should He^+ and O^+ whistlers be detected by future VLF experiments we expect that a technique similar to that discussed in this paper could be applied to them to obtain the He^+ and O^+ number densities.

Acknowledgments. We would like to express our appreciation to Professor J. A. Van Allen for his interest in this study and to Professor N. M. Brice for his helpful discussions.

The research at the University of Iowa was supported in part by the Office of Naval Research under contract Nonr 1509(06).

References

Beghin, C., and C. Siredey, Un procédé d'analyse fine des sifflements atmosphériques, *Ann. Geophys., 20,* 301–308, 1964.

Gurnett, D. A., Ion cyclotron whistlers, Ph.D. dissertation, Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa (*Res. Rept. 65-2*), 1965.

Gurnett, D. A., and N. M. Brice, Cyclotron damping of ion cyclotron whistlers; a method for the determination of ion temperature, Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa (*Res. Rept. 65-34*), 1965.

Gurnett, D. A., S. D. Shawhan, N. M. Brice, and R. L. Smith, Ion cyclotron whistlers, *J. Geophys. Res., 70,* 1665–1688, 1965.

Jensen, D. C., and J. C. Cain, An interim geomagnetic field (abstract), *J. Geophys. Res., 67,* 3568–3569, 1962.

Lacey, L. L., *Statistical Methods in Experimentation,* The Macmillan Company, New York, 1959.

Shawhan, S. D., Experimental observations of proton whistlers from the Injun 3 VLF data, *J. Geophys. Res., 71,* 29–44, 1966.

Shawhan, S. D., and D. A. Gurnett, Fractional concentration of hydrogen ions in the ionosphere from VLF proton whistler measurement, *J. Geophys. Res., 71,* 46–59, 1966.

Smith, R. L., N. M. Brice, J. Katsufrakus, D. A. Gurnett, S. D. Shawhan, J. S. Belrose, and R. E. Barrington, An ion gyrofrequency phenomenon observed in satellites, *Nature, 204,* 274–275, October 17, 1964.

Stix, T. H., *The Theory of Plasma Waves,* McGraw-Hill Book Company, New York, 1962.

Storey, L. R. O., An investigation of whistling atmospherics, *Phil. Trans. Roy. Soc. London, A, 246,* 113–141, 1953.

(Manuscript received September 17, 1965.)